

Given $y(t) = x(t) * h(t)$, we want $y(t) = \sum_k (\dots) e^{jk\omega_0 t}$, what is $(\dots)$?

Consider $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$, a_k is a scaler, then

$$y(t) = x(t) * h(t) = \left(\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \right) * h(t) = \sum_{k=-\infty}^{\infty} a_k (e^{jk\omega_0 t} * h(t)).$$

For single $e^{jk\omega_0 t} * h(t)$, we have

$$e^{jk\omega_0 t} * h(t) = \int_{-\infty}^{\infty} h(\tau) e^{jk\omega_0(t-\tau)} d\tau = e^{jk\omega_0 t} \int_{-\infty}^{\infty} h(\tau) e^{-jk\omega_0 \tau} d\tau.$$

Let $H(j\omega_k) = \int_{-\infty}^{\infty} h(\tau) e^{-jk\omega_0 \tau} d\tau$, then

$$y(t) = \sum_{k=-\infty}^{\infty} a_k (e^{jk\omega_0 t} * h(t)) = \sum_{k=-\infty}^{\infty} a_k H(j\omega_k) e^{jk\omega_0 t}.$$

For a given $h(t)$, $H(j\omega_k)$ is easy to calculate.

So the problem becomes: **when there is a $x(t)$, how to find its a_k for each $e^{jk\omega_0 t}$?**

Consider multiplying $x(t)$ by $e^{-jn\omega_0 t}$, we have

$$x(t) e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t}.$$

Then

$$\int_0^T x(t) e^{-jn\omega_0 t} dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{j(k-n)\omega_0 t} dt = \sum_{k=-\infty}^{\infty} a_k \left[\int_0^T e^{j(k-n)\omega_0 t} dt \right].$$

For single $\int_0^T e^{j(k-n)\omega_0 t} dt$, we have

$$\int_0^T e^{j(k-n)\omega_0 t} dt = \int_0^T \cos((k-n)\omega_0 t) dt + j \int_0^T \sin((k-n)\omega_0 t) dt = \begin{cases} T, & k = n \\ 0, & k \neq n \end{cases}.$$

Thus,

$$\int_0^T x(t) e^{-jn\omega_0 t} dt = T a_n \implies a_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} dt \implies \boxed{a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt}.$$

Now a_k and $H(j\omega_k)$ are both computable, then we can find $y(t)$ by $\sum_{k=-\infty}^{\infty} a_k H(j\omega_k) e^{jk\omega_0 t}$.

When $x(t)$ is real,

$$\boxed{a_k = a_{-k}^*}.$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = a_0 + \sum_{k=1}^{\infty} 2|a_k| \cos(k\omega_0 t + \arg(a_k)).$$

When $x(t)$ is odd,

$$\begin{aligned} a_{-k} &= \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{jk\omega_0 t} dt \\ &= \frac{1}{T} \int_{T/2}^{-T/2} x(-t') e^{-jk\omega_0 t'} d(-t') \\ &= \frac{-1}{T} \int_{-T/2}^{T/2} x(t') e^{-jk\omega_0 t'} dt' \\ &= -a_k. \end{aligned}$$

So

$$\operatorname{Re}(a_k) = \operatorname{Re}(a_{-k}) = 0.$$

When $x(t)$ is even,

$$\begin{aligned} a_{-k} &= \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{jk\omega_0 t} dt \\ &= \frac{1}{T} \int_{T/2}^{-T/2} x(-t') e^{-jk\omega_0 t'} d(-t') \\ &= \frac{1}{T} \int_{-T/2}^{T/2} x(t') e^{-jk\omega_0 t'} dt' \\ &= a_k. \end{aligned}$$

So

$$\operatorname{Im}(a_k) = \operatorname{Im}(a_{-k}) = 0.$$

	CT	DT
angular freq	$\omega_0 = 2\pi/T_0$	$\Omega_0 = 2\pi/N_0$
FS rep of x	$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	$x[n] = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 n}$
FS rep of y	$y(t) = \sum_{k=-\infty}^{\infty} a_k H(j\omega_k) e^{jk\omega_0 t}$	$y[n] = \sum_{k=-\infty}^{\infty} a_k H(j\omega_k) e^{jk\Omega_0 n}$
FS response	$H(j\omega_k) = \int_{-\infty}^{\infty} h(\tau) e^{-jk\omega_0 \tau} d\tau$	$H(j\omega_k) = \sum_{n=-\infty}^{\infty} h[n] e^{-jk\Omega_0 n}$
coefficients of x	$a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt$	$a_k = \frac{1}{T} \sum_T x[n] e^{-jk\Omega_0 n}$
coefficients of y	$b_k = a_k H(j\omega_k)$	$b_k = a_k H(j\omega_k)$

Table 1. Summary & Comparison