

Question 1. Let $\mathcal{P} = \{p_1, p_2, p_3, \dots\} = \{3, 5, 7, \dots\}$ be the set of odd prime numbers in increasing order. For fixed $k \in \mathbb{Z}_{\geq 2}$ and $\delta \in \mathbb{Q}$, define a sequence $\{X_n\}_{n=1}^{\infty}$ as:

$$X_n = \frac{1}{k} \sum_{i=0}^{k-1} p_{n+i} + \delta$$

Let $\mathcal{A} = \{X_n \in \mathbb{Z}, n \in \mathbb{Z}^+\}$ be the set of all integer terms of the sequence.

Prove the following:

1. If $\delta = 0$ and $k = 2$, then $\mathcal{A} \cap \mathcal{P} = \emptyset$.
2. If $\delta = 0$ and k is odd, then $\mathcal{A} \cap \mathcal{P} \neq \emptyset$.
3. For any $\delta \in \mathbb{Q}$ and $k > 2$ such that $\mathcal{A}$ contains infinitely many odd integers, $\mathcal{A} \cap \mathcal{P} \neq \emptyset$.