

Question. Why *normal equation* $A^\top A \tilde{x} = A^\top b$ is always solvable?

Proof. Noticed that

$$\text{rank}(A) = \text{rank}(A^\top A) \leq \text{rank}\left(\begin{bmatrix} A^\top A & A^\top b \end{bmatrix}\right) = \text{rank}\left(A^\top \begin{bmatrix} A & b \end{bmatrix}\right) \leq \text{rank}(A^\top) = \text{rank}(A),$$

then every inequality signs above is equality, so

$$\text{rank}(A^\top A) = \text{rank}\left(\begin{bmatrix} A^\top A & A^\top b \end{bmatrix}\right) \implies A^\top b \in \mathcal{C}(A^\top A).$$

Then it's clear that $A^\top A \tilde{x} = A^\top b$ is solvable. □

If the columns of A are linearly independent ($A^\top A$ is invertible), then

$$\tilde{x} = (A^\top A)^{-1} A^\top b.$$