

An incredible integral

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Question 1. Compute the follwing integral:

$$\frac{\int_0^\infty e^{-s} s^5 ds}{2} + \frac{\int_{-\infty}^\infty e^{-\frac{t^2}{2}} dt}{\int_0^\infty \sin t^2 dt} \times \left[\frac{\sum_{n=0}^\infty \frac{(-1)^n}{2n+1}}{\int_0^\infty \frac{\sin x}{x} dx} + \frac{\sum_{n=1}^\infty \arctan \frac{2}{n^2}}{\lim_{t \rightarrow 0^+} \left(\int_{-2026}^{2026} \frac{t \cos x}{x^2 + t^2} dx \right)} \right]$$

$$\frac{\lim_{n \rightarrow \infty} \left\{ \left[\left(\int_0^1 \frac{x^{n-1}}{1+x} dx \right) n - \frac{1}{2} \right] \frac{n}{2} \right\}}{H}$$

Answer. We break the integral expression into 8 part:

$$\frac{\frac{A}{2} + \frac{B}{C} \times \left[\frac{D}{E} + \frac{F}{G} \right]}{H}$$

For 1st part,

$$A = \int_0^\infty e^{-s} s^5 ds$$

Definition 1. The definition of Γ function:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \quad (1)$$

Lemma 1. For $\forall n \in \mathbb{N}$, we have

$$\Gamma(n+1) = n! \quad (2)$$

Proof. By calculation,

$$\int_0^\infty x^{\alpha-1} e^{-x} dx = [-x^{\alpha-1} e^{-x}]_0^\infty + (\alpha-1) \int_0^\infty x^{\alpha-2} e^{-x} dx = (\alpha-1) \times \Gamma(\alpha-1)$$

We notice that,

$$\Gamma(\alpha) = (\alpha-1) \times \Gamma(\alpha-1)$$

Let $\alpha = 1$ in (1)

$$\Gamma(1) = \int_0^\infty e^{-x} dx = [-e^{-x}]_0^\infty = 1$$

So for $\forall n \in \mathbb{N}$,

$$\Gamma(n+1) = n \times \Gamma(n) = \cdots = n! \times \Gamma(1) = n!$$

□

Then,

$$A = \int_0^\infty e^{-s} s^5 ds \stackrel{(1)}{=} \Gamma(6) \stackrel{(2)}{=} 5! = 120$$

For 2nd part,

$$B = \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt$$

By substitution of $u = \frac{t}{\sqrt{2}}$,

$$\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}} dt = \sqrt{2} \int_{-\infty}^{\infty} e^{-u^2} du$$

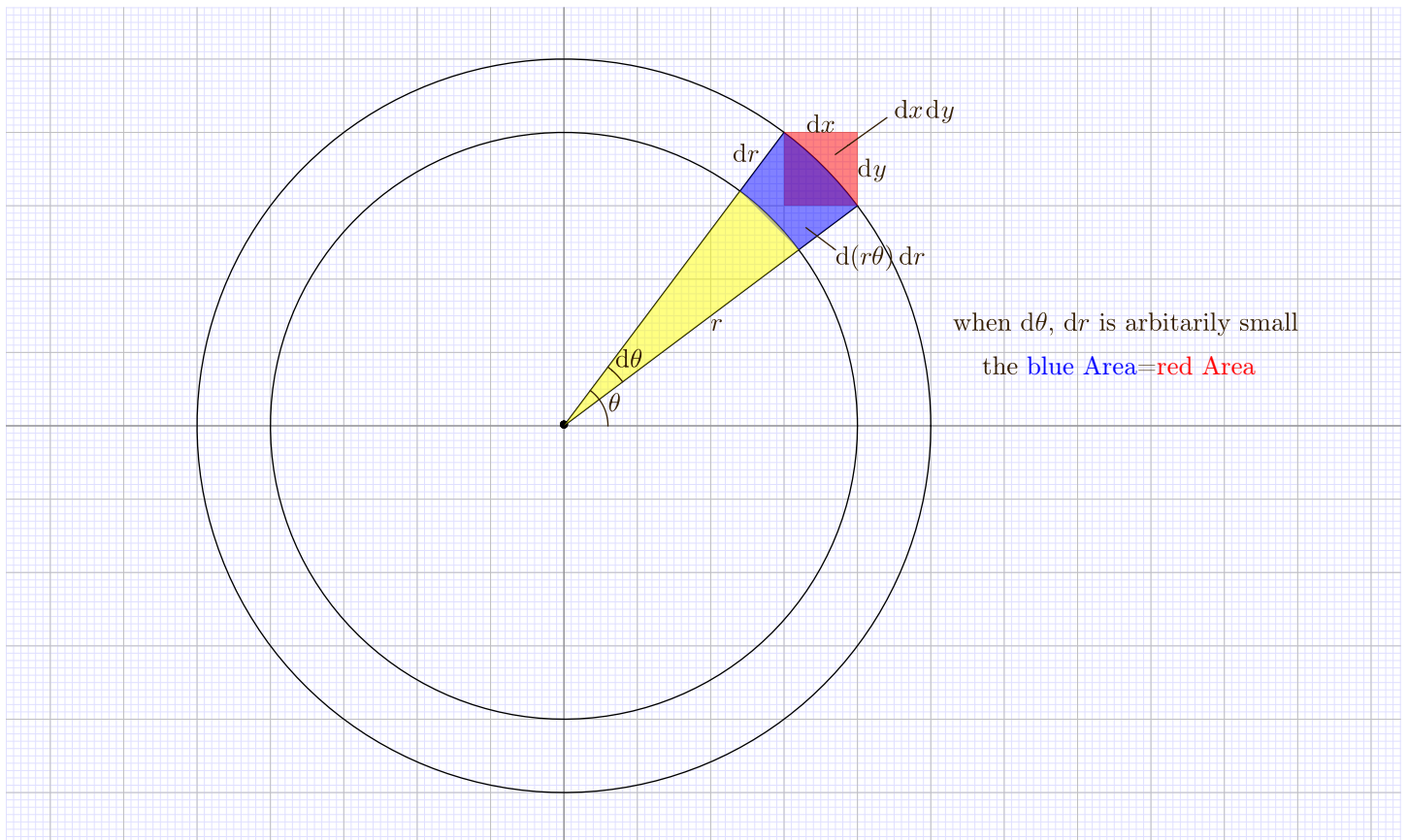
Lemma 2. *The Gaussian integral:*

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \quad (3)$$

Proof. Let $I = \int_{-\infty}^{\infty} e^{-x^2} dx$,

$$I^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2} dy \right) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy$$

Let $x = r \cos \theta$, $y = r \sin \theta$, from geometry perspective:



$$dx dy = dr d(r\theta) = r dr d\theta$$

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy = \int_0^{2\pi} d\theta \int_0^{\infty} e^{-r^2} r dr = 2\pi \left(-\frac{1}{2} \right) [e^{-r^2}]_0^{\infty} = \pi$$

□

Then,

$$B = \sqrt{2} I \stackrel{(3)}{=} \sqrt{2\pi}$$

For 3rd part,

$$C = \int_0^{\infty} \sin t^2 dt$$

By substitution of $t = \sqrt{x}$,

$$\int_0^\infty \sin t^2 dt = \frac{1}{2} \int_0^\infty \frac{\sin x}{\sqrt{x}} dx$$

From (3), we know that:

$$\frac{\sqrt{\pi}}{2} = \int_0^\infty e^{-u^2} du = \sqrt{t} \int_0^\infty e^{-tx^2} dx \implies \frac{1}{\sqrt{t}} = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-tx^2} dx \quad (4)$$

Then,

$$\begin{aligned} C &= \frac{1}{2} \int_0^\infty \sin x \left(\frac{2}{\sqrt{\pi}} \int_0^\infty e^{-xu^2} du \right) dx \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty \left(\int_0^\infty e^{-xu^2} du \right) \sin x dx \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty \left(\int_0^\infty e^{-xu^2} \sin x dx \right) du \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty \left[\frac{-e^{-ax}}{a^2+1} (a \sin x + \cos x) \right]_0^\infty du \\ &= \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{u^4+1} du \end{aligned} \quad (5)$$

$$\begin{aligned} &= \frac{1}{2\sqrt{\pi}} \left(\int_0^\infty \frac{u^2+1}{u^4+1} du - \int_0^\infty \frac{u^2-1}{u^4+1} du \right) \\ &= \frac{1}{2\sqrt{\pi}} \left(\int_0^\infty \frac{1+\frac{1}{u^2}}{u^2+\frac{1}{u^2}} du - \int_0^\infty \frac{1-\frac{1}{u^2}}{u^2+\frac{1}{u^2}} du \right) \\ &= \frac{1}{2\sqrt{\pi}} \left(\int_0^\infty \frac{d\left(u-\frac{1}{u}\right)}{\left(u-\frac{1}{u}\right)^2+2} - \int_0^\infty \frac{d\left(u+\frac{1}{u}\right)}{\left(u+\frac{1}{u}\right)^2-2} \right) \\ &= \frac{1}{2\sqrt{\pi}} \left(\int_{-\infty}^\infty \frac{dt}{t^2+2} - \int_0^\infty \frac{dt}{t^2-2} \right) \\ &= \frac{1}{2\sqrt{2\pi}} \left\{ \left[\arctan \frac{t}{\sqrt{2}} \right]_{-\infty}^\infty - \left[\ln \left(\frac{t-\sqrt{2}}{t+\sqrt{2}} \right) \right]_{-\infty}^\infty \right\} \\ &= \frac{1}{2\sqrt{2\pi}} (\pi - 0) \\ &= \sqrt{\frac{\pi}{8}} \end{aligned} \quad (6)$$

The process in (5) and (6) requires a little bit tricks, but makes sence.

For 4th part,

$$D = \sum_{n=0}^\infty \frac{(-1)^n}{2n+1}$$

By taylor series of $\arctan x$, for $|x| \leq 1$ we have

$$\arctan x = \sum_{k=0}^\infty \frac{(-1)^k x^{2k+1}}{2k+1} \quad (7)$$

Then,

$$D \stackrel{(7)}{=} \arctan 1 = \frac{\pi}{4}$$

For 5th part,

$$E = \int_0^\infty \frac{\sin x}{x} dx$$

Let $f(x) = \sin x$, by Laplace transform, we have

$$F(s) = \mathcal{L}[\sin x] = \frac{1}{s^2+1}$$

So,

$$\mathcal{L}\left[\frac{\sin x}{x}\right] = \int_s^\infty F(u) du = \int_0^\infty \frac{1}{u^2+1} du \quad (8)$$

And,

$$\int_0^\infty \frac{1}{u^2+1} du = [\arctan u]_s^\infty = \frac{\pi}{2} - \arctan s$$

Then,

$$\int_0^\infty \frac{\sin x}{x} dx = \lim_{s \rightarrow 0} \mathcal{L} \left[\frac{\sin x}{x} \right] \stackrel{(8)}{=} \lim_{s \rightarrow 0} \left(\frac{\pi}{2} - \arctan s \right)$$

$$E = \frac{\pi}{2}$$

For 6th part,

$$F = \sum_{n=1}^{\infty} \arctan \frac{2}{n^2}$$

We know that,

$$\arctan x + \arctan y = \arctan \frac{x+y}{1-xy}$$

By observation,

$$\arctan \frac{2}{n^2} = \arctan \frac{(1+n) + (1-n)}{1 - (1+n)(1-n)} = \arctan(1+n) + \arctan(1-n) \quad (9)$$

Then,

$$\sum_{n=1}^{\infty} \arctan \frac{2}{n^2} \stackrel{(9)}{=} \sum_{n=1}^{\infty} [\arctan(1+n) + \arctan(1-n)] = \sum_{n=1}^{\infty} [\arctan(1+n) - \arctan(n-1)]$$

Sum up them together,

$$F = \sum_{n=1}^{\infty} (\arctan 2 - \arctan 0 + \arctan 3 - \arctan 1 + \dots + \arctan(\infty)) = \frac{3}{4}\pi$$

For 7th part,

$$G = \lim_{t \rightarrow 0^+} \left(\int_{-2026}^{2026} \frac{t \cos x}{x^2 + t^2} dx \right)$$

By substitution of $u = \frac{x}{t}$,

$$\int_{-2026}^{2026} \frac{t \cos x}{x^2 + t^2} dx = \int_{-2026/t}^{2026/t} \frac{t^2 \cos(tu)}{t^2(u^2 + 1)} du = \int_{-2026/t}^{2026/t} \frac{\cos(tu)}{u^2 + 1} du$$

And $t \rightarrow 0^+$,

$$\int_{-2026/t}^{2026/t} \frac{\cos(tu)}{u^2 + 1} du = \int_{-\infty}^{\infty} \frac{1}{u^2 + 1} du = [\arctan u]_{-\infty}^{\infty} = \pi$$

Then,

$$G = \pi$$

For 8th part,

$$H = \lim_{n \rightarrow \infty} \left\{ \left[\left(\int_0^1 \frac{x^{n-1}}{1+x} dx \right) n - \frac{1}{2} \right] \frac{n}{2} \right\}$$

By calculation,

$$\int_0^1 \frac{x^{n-1}}{1+x} dx = \frac{1}{n} \left(\left[\frac{x^n}{1+x} \right]_0^1 + \int_0^1 \frac{x^n}{(1+x)^2} dx \right) = \frac{1}{n} \left(\frac{1}{2} + \int_0^1 \frac{x^n}{(1+x)^2} dx \right)$$

And,

$$\begin{aligned} \int_0^1 \frac{x^n}{(1+x)^2} dx &= \frac{1}{n+1} \left(\frac{1}{4} + \int_0^1 \frac{x^{n+1}}{(1+x)^3} dx \right) \\ &= \frac{1}{4(n+1)} + \frac{1}{(n+1)(n+2)} \left(\frac{1}{8} + \int_0^1 \frac{x^{n+1}}{(1+x)^4} dx \right) \end{aligned}$$

By simplification,

$$H = \lim_{n \rightarrow \infty} \left(\frac{n}{8(n+1)} + \frac{1}{2 \left(1 + \frac{1}{n} \right) (n+2)} \left(\frac{1}{8} + \int_0^1 \frac{x^{n+1}}{(1+x)^4} dx \right) \right)$$

Since $n \rightarrow \infty$,

$$\frac{n}{8(n+1)} \rightarrow \frac{1}{8}$$

$$\frac{1}{2\left(1+\frac{1}{n}\right)(n+2)} \rightarrow 0$$

Then,

$$H=\frac{1}{8}$$

To sum up,

$$\frac{\frac{A}{2}+\frac{B}{C}\times\left[\frac{D}{E}+\frac{F}{G}\right]}{H}=\frac{\frac{120}{2}+\frac{\sqrt{2\pi}}{\sqrt{\frac{\pi}{8}}}\times\left[\frac{\frac{\pi}{4}}{2}+\frac{\frac{3}{4}\pi}{\pi}\right]}{\frac{1}{8}}=\textcolor{red}{520}$$

Appendix A Useful shortcuts

Symbol	Shorcut
$\int$	Shift+I Tab
∞	@@
$\sum$	Shift+S Tab
$\frac{1}{2}$	Alt F

Table 1.